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THE MOMENT OF INERTIA METHOD TO CALCULATE EQUIVALENT RANGES IN NON-PROPORTIONAL MULTIAXIAL HISTORIES[1]

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Abstract

A critical issue in multiaxial damage calculation in non-proportional (NP) histories is to find the equivalent stress or strain ranges and mean components associated with each rainflow-counted cycle of the stress (or strain) path. A traditional way to find such ranges is to use enclosing surface methods, which search for convex enclosures, such as balls or prisms, of the entire history path in stress or strain diagrams. These methods only work for relatively simple load histories, since the enclosing surfaces lose information of the original history. This work presents a new approach to evaluate equivalent stress and strain ranges in NP histories, called the Moment Of Inertia (MOI) method. It is an integral approach which assumes instead that the path contour in the stress diagram is a homogeneous wire with a unit mass. The center of mass of such wire gives then the mean component of the path, while the moments of inertia of the wire can be used to obtain the equivalent stress or strain ranges. Experimental results for 13 different multiaxial histories prove the effectiveness of the MOI method to predict fatigue lives.

Key words: Multiaxial fatigue; Non-proportional loading; Equivalent stress range.

MÉTODO DO MOMENTO DE INÉRCIA PARA CALCULAR AMPLITUDES EQUIVALENTES EM HISTÓRIAS MULTIAXIAIS NÃO-PROPORCIONAIS

Resumo

Uma etapa crítica para a previsão de vida em histórias multiaxiais não-proporcionais (NP), envolve o cálculo das componentes média e alternada, equivalentes de uma história complexa. Um método tradicional consiste no uso de superfícies envoltórias (como bolas ou prismas) que contenham o caminho de um carregamento traçado em um diagrama de tensões. Esse método somente é eficiente em carregamentos relativamente simples, uma vez que essas superfícies perdem informação sobre a história original. Este trabalho apresenta um método para cálculo de tensões equivalentes em histórias NP, chamado de método do Momento De Inércia (MDI), baseado em uma formulação integral que assume que o caminho do carregamento no diagrama de tensões é um arame homogêneo com massa unitária. O seu centro de massa define a componente média do carregamento, enquanto que seus momentos de inércia de rotação são usados para calcular as amplitudes equivalentes. Resultados experimentais para 13 histórias multiaxiais provam a eficácia do método proposto.

Palavras-chave: Fadiga multiaxial; Carregamento não-proporcional; Gama de tensão equivalente.

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1 INTRODUCTION

Multiaxial fatigue lives can be calculated from equivalent stress (or strain) ranges and their mean components.(1) However, estimating such ranges and mean values for non-proportional (NP) loading cycles is not an easy task. These components are traditionally estimated by convex circular, ellipsoidal or prismatic enclosures of the entire history path in stress or strain diagrams. However, enclosing surface methods are not suited for complex NP histories, since they do not account for path shape dependence of fatigue damage.

Consider a periodic load history formed by repeatedly following a given loading path domain $\mathcal{D}$, where $\mathcal{D}$ contains all points from the stress or strain variations along one of its periods. Assume that two out-of-phase shear stresses τ_B and τ_{B2} act parallel to the critical plane, where the crack will most likely initiate. Both τ_B and τ_{B2} influence the growth of shear cracks along the critical plane. To calculate the maximum shear stress range $\Delta\tau_{max}$ at the critical plane, it is necessary to draw the path $\mathcal{D}$ of the stress history along a $\tau_B \times \tau_{B2}$ diagram (Figure 1).

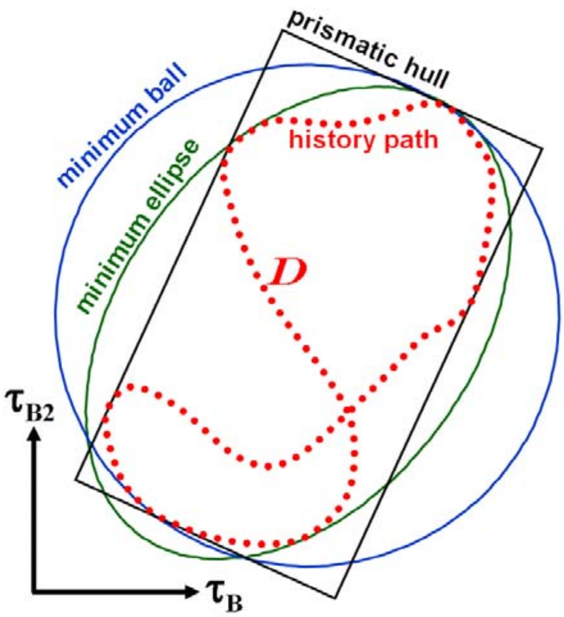

Figure 1. Periodic (or single) stress history path D in a $\tau_B \times \tau_{B2}$ diagram, enclosed in surfaces such as circles (balls), ellipses and rectangular prisms.

The search for an effective range using the deviatoric stress path started with the pioneering work of Dang Van,(2) who studied various methods to define and calculate it. Since then, several "enclosing surface methods" have been proposed,(3-7) which try to find circles, ellipses or rectangles that contain the entire load path (in the 2D case). In a nutshell, in the 2D case, the Minimum Ball (MB) method(3) searches for the circle with minimum radius that contains $\mathcal{D}$; the minimum ellipse (ME) methods(4-6) search for an ellipse with semi-axes a and b that contains $\mathcal{D}$ with minimum area $\pi{\cdot}a{\cdot}b$ or minimum norm $(a^2 + b^2)^{1/2}$; and the maximum prismatic hull (MPH) methods(5,7) search among the smallest rectangles that contain $\mathcal{D}$ the one with maximum area or maximum diagonal (it's a max-min search problem). The value of $\Delta\tau_{max}$ in Figure 1 would either be assumed as the value of the circle diameter, or twice the ellipse norm, or the length of the enclosing rectangle diagonal. If the history path was

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represented in a $\gamma_B \times \gamma_{B2}$ shear strain diagram, these exact same methods would result in estimates for the maximum shear strain range $\Delta\gamma_{max}$.
The enclosing surface methods can also be applied to traction-torsion load histories, if a $\sigma_x \times \tau_{xy}\sqrt{3}$ diagram is considered. The effective range in this case is the Mises stress range $\Delta\sigma_{Mises}$. Similarly, for traction-torsion histories where plastic strains dominate, a strain diagram $\varepsilon_x \times \gamma_{xy}/\sqrt{3}$ can be used to predict an effective Mises strain range $\Delta\varepsilon_{Mises}$.
Such enclosing surface methods can be extended to histories involving more than two stress or strain components. E.g., if the history path is plotted in a 3D diagram representing 3 stress or strain components, the enclosing surface methods will search for spheres, ellipsoids or rectangular prisms. For higher dimension diagrams, the search is for hyperspheres, hyperellipsoids, and rectangular hyperprisms. However, this practice can lead to significant errors, since each enclosing surface will reflect an effective range calculated on different planes at different points in time.(1)
The recommended approach for general 6D histories involving all stress (or strain) components is the one proposed by Bannantine and Socie:(1) to project them onto Case A and Case B candidate planes,(1) resulting for the Case B planes in searches for effective ranges in 2D diagrams $\tau_B \times \tau_{B2}$ or $\gamma_B \times \gamma_{B2}$.
Enclosing surface methods can be useful to estimate the equivalent stress (or strain) amplitude associated with NP loading paths. However, such methods have three issues. First, among all enclosing surface methods, only the MB has a physical foundation. The search of the minimum ball enclosing a history path in the deviatoric space corresponds to the search of the elastic-shakedown state that the material grains in the neighbor of the point of interest could attain under periodic loading, considering an isotropic and/or kinematic hardening behavior.(1) In other words, fatigue crack initiation is avoided if an elastic shakedown state can be reached. On the other hand, enclosing ellipsoids and prisms are not derived from physical considerations, they are empirical methods that try to interpolate the limit cases between a proportional loading history and a highly non-proportional one.(1) Even so, these methods still have their practical value as engineering tools for relatively simple loading paths, as long as their effectiveness is experimentally verified.
The second issue is that each portion of the considered path should not involve more than 1 cycle. Otherwise, if it is considered as a single cycle, the actual damage might be underestimated. Instead, a multiaxial rainflow algorithm should be applied to the entire stress or strain history, and then an enclosing surface method should be applied for the path of each rainflow-counted reversal.
Finally, the third issue involves information loss. Enclosing surface algorithms do not take into account the actual loading path, but only the convex hulls associated with them. For instance, consider a square path ABCD in a 2D deviatoric space. The convex hull of such path, defined as the convex enclosure with minimum area that contains the entire path, is the square itself. An hourglass-shaped path ABDC or ADBC would have the same convex hull: the square ABCD. It is not difficult to prove that the enclosing circle, ellipse or prismatic hull from any existing enclosing surface method would result in the same enclosure for these 3 considered paths, treating them as identical. In general, all paths that share the same convex hull share as well the same enclosing surface for a given method, even though they might lead to different equivalent amplitudes and fatigue lives. This third issue is addressed by a novel method to calculate equivalent and mean components that takes into account the actual loading path, not only its convex hull. This new method is presented next.

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2 THE MOMENT OF INERTIA (MOI) METHOD

The Moment Of Inertia (MOI) method is proposed here to calculate alternate and mean components of complex NP load histories. To accomplish that, the history must first be represented in a 2D subspace of the transformed 5D Euclidean stress or strain space. The MOI method assumes that the 2D path/domain $\mathcal{D}$, represented by a series of points *(X, Y)* from the stress or strain variations along it, is analogous to a homogeneous wire with unit mass. Note that *X* and *Y* can have stress or strain units, but they are completely unrelated to the directions *x* and *y* usually associated with the material surface. The mean component of $\mathcal{D}$ is assumed, in the MOI method, to be located at the center of gravity of this imaginary homogeneous wire shaped as the history path. Such center of gravity is located at the perimeter centroid (X_c, Y_c) of $\mathcal{D}$, calculated from contour integrals along the entire path.

$$X_c = \frac{1}{p} \cdot \oint X \cdot dp, \quad Y_c = \frac{1}{p} \cdot \oint Y \cdot dp, \quad p = \oint dp \tag{1}$$

Where *dp* is the length of an infinitesimal arc of $\mathcal{D}$ and *p* is the path perimeter (Figure 2).

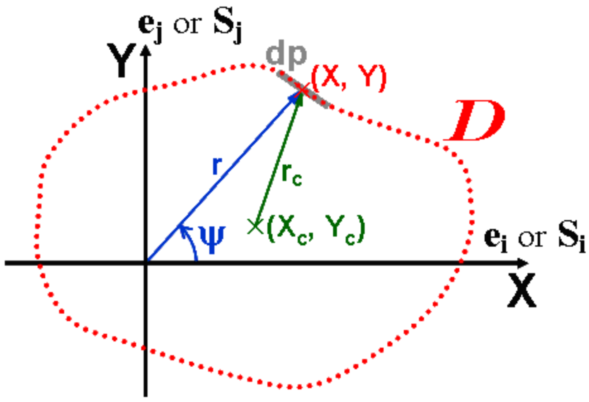

Figure 2. Load history path, assumed as a homogeneous wire with unit mass in the deviatoric 2D space.

The MOI method is so called because it makes use of the mass moments of inertia (MOI) of such homogeneous wire. These moments are first calculated with respect to the origin *O* of the diagram, assuming the wire has unit mass, resulting in:

$$I^O_{XX} = \frac{1}{p} \cdot \oint Y^2 \cdot dp, \quad I^O_{YY} = \frac{1}{p} \cdot \oint X^2 \cdot dp, \quad I^O_{XY} = -\frac{1}{p} \cdot \oint X \cdot Y \cdot dp \tag{2}$$

Then, the mass moments of inertia of such unit mass wire, with respect to its center of gravity (X_c, Y_c), are obtained. They are computed from the moments of inertia of the path $\mathcal{D}$ with respect to its perimeter centroid (X_c, Y_c), which are easily obtained from the parallel axis theorem, assuming a unit mass:

$$I_{XX} = I^O_{XX} - {Y_c}^2, \quad I_{YY} = I^O_{YY} - {X_c}^2, \quad I_{XY} = I^O_{XY} + X_c \cdot Y_c \tag{3}$$

The MOI method simply assumes that the deviatoric stress or strain ranges, $\Delta S \equiv \Delta\sigma_{Mises}$ or $\Delta e \equiv \Delta\varepsilon_{Mises}$, depend on the mass moment of inertia I_{ZZ} with respect to the perimeter centroid, perpendicular to the *X*-*Y* plane. This is physically sound, since history paths further away from their perimeter centroid PC will contribute more to the effective range and amplitude, which is coherent with the fact that wire segments further away from the center of gravity of an imaginary homogeneous wire contribute more to its MOI. Note that the use of integral parameters to evaluate NP paths is not new, it was already used e.g. in Kida et al.[8] to estimate the non-proportionality factor. But the use of a moment of inertia analogy to obtain effective ranges is a novel idea, a true alternative for the existing enclosing surface methods.

From the perpendicular axis theorem, which states that $I_{ZZ} = I_{XX} + I_{YY}$, and from a dimensional analysis, it is found that:

$$\frac{\Delta\sigma_{Mises}}{2} \text{ or } \frac{\Delta\varepsilon_{Mises}}{2} = \sqrt{3 \cdot I_{ZZ}} = \sqrt{3 \cdot (I_{XX} + I_{YY})} \tag{4}$$

The factor $\sqrt{3}$ is introduced to guarantee that a proportional loading path, represented by a straight segment with length L, perimeter $2L$ and unit mass $m = 1$, will result in the expected range $\Delta\sigma_{Mises}$ or $\Delta\varepsilon_{Mises}$ equal to L (since the MOI of a straight wire with respect to its centroid is $m \cdot L^2/12$).

Note that the above definitions are coherent, since they are independent of the X-Y system orientation because since $I_{XX} + I_{YY}$ is an invariant, it is equal to the sum of the principal MOI $I_1 + I_2$ of the homogeneous wire which represents the loading path.

The MOI method is simple to calculate, in special for polygonal load histories. The mass moments of inertia of curved histories are also easy to calculate from fine polygonal discretizations. In addition, the MOI method can make use of classical mass moment of inertia tables, or even CAD programs applied to arbitrarily-shaped homogeneous wires, to calculate I_{XX}, I_{YY}, I_{XY} and I_{ZZ}.

To use the MOI approach in polygonal load history paths, it is enough to combine the expression for the moment of inertia of an inclined straight wire and the parallel axis theorem. If each side i of the polygon has length Δp_i, centered at (X_{ci}, Y_{ci}), and making an angle ψ_i with respect to the horizontal (Figure 3), then the load path perimeter centroid PC and the MOI expressions with respect to the origin are obtained from Equation 5.

$$p = \sum_i \Delta p_i, \quad X_c = \frac{1}{p} \cdot \sum_i X_{ci} \cdot \Delta p_i, \quad Y_c = \frac{1}{p} \cdot \sum_i Y_{ci} \cdot \Delta p_i$$

$$I_{XX}^O = \frac{1}{p} \cdot \sum_i \left(\frac{\Delta p_i^2}{12} \sin^2 \psi_i + Y_{ci}^2 \right) \cdot \Delta p_i, \quad I_{YY}^O = \frac{1}{p} \cdot \sum_i \left(\frac{\Delta p_i^2}{12} \cos^2 \psi_i + X_{ci}^2 \right) \cdot \Delta p_i,$$

$$I_{XY}^O = -\frac{1}{p} \cdot \sum_i \left(\frac{\Delta p_i^2}{12} \sin \psi_i \cos \psi_i + X_{ci} Y_{ci} \right) \cdot \Delta p_i \tag{5}$$

The MOI with respect to the load path PC is then calculated from Equation 3. In the next section, the MOI and enclosing surface methods are compared and tested against comprehensive experimental data gathered from the literature.

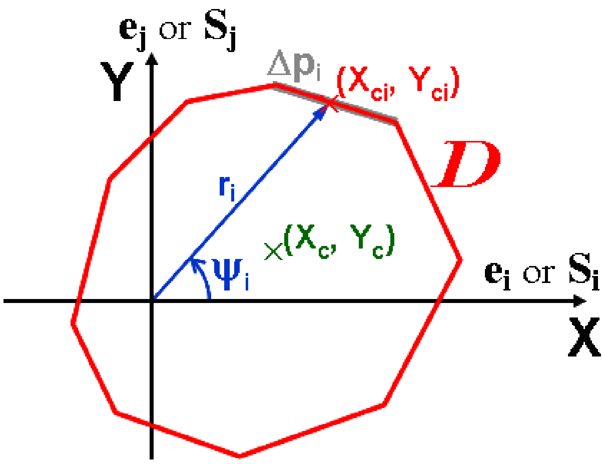

Figure 3. Application of the MOI method to polygonal history paths.

3 EXPERIMENTAL EVALUATION OF THE MOI AND ENCLOSING SURFACE FATIGUE LIFE PREDICTIONS

The MOI and enclosing surface estimates of effective ranges are now used to reproduce the multiaxial fatigue lives of 304 stainless steel specimens tested by Kida et al.[8] Thirteen periodic histories are studied, represented by the block loadings shown in Figure 4 for Cases 0 through 12. The multiaxial fatigue lives are calculated using the Smith-Watson-Topper (SWT) model in Bannantine-Socie's critical plane approach,[1] searching for the plane where the damage parameter $\sigma_{max}\cdot\Delta\varepsilon/2$ is maximized. The material properties used in these calculations are:

$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{1754}\right)^{1/0.276}$$

$$(\sigma_{max}\frac{\Delta\varepsilon}{2})_{max} = \frac{757^2}{E}(2N)^{2b} + 30.5\cdot(2N)^{b+c}$$

$$E = 197{,}000\,MPa, \quad b = -0.0886, \quad c = -0.303 \qquad (6)$$

Table 1 shows the experimental fatigue lives and the associated MOI, MB and MPH method life predictions for each one of the 13 loading histories. Note that the MOI method considers 2 cycles per load block for Cases 1 through 4; this number of cycles can be deterministically obtained using the Modified Wang-Brown rainflow algorithm described in Meggiolaro and Castro.[9]

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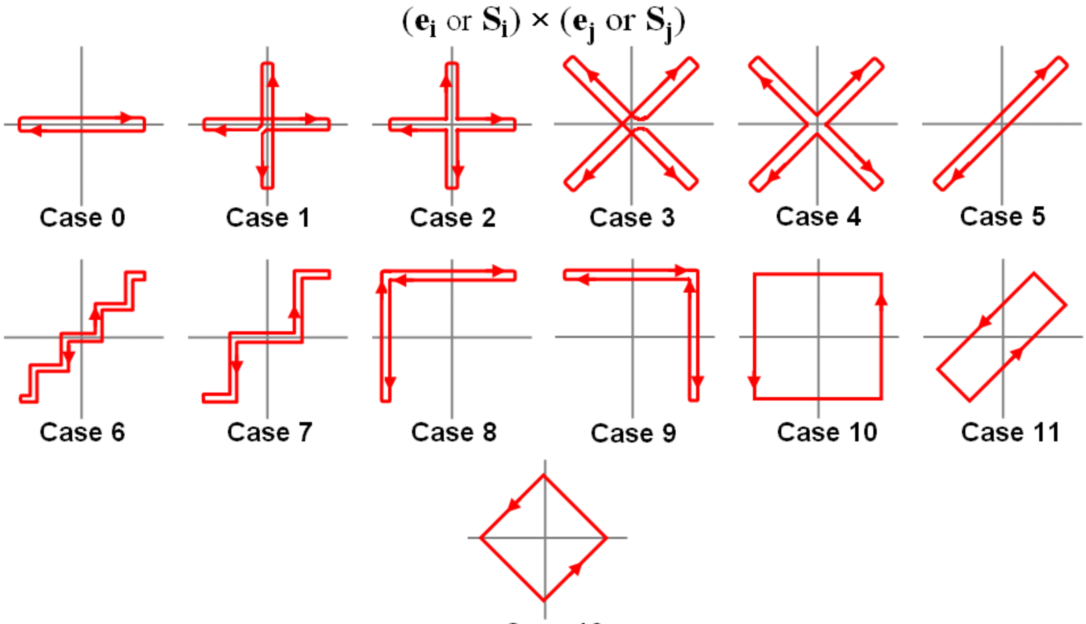

Figure 4. History paths used in the experimental validation of the equivalent range predictions.

Table 1. Fatigue life *N* (in cycles) experimentally measured and predicted using the Smith-Watson-Topper damage model and the Moment Of Inertia (MOI), Minimum Ball (MB) and Maximum Prismatic Hull (MPH) methods. Note that Cases 1-4 consider 2 cycles per block (e.g. the measured life for Case 1 was 1,400 loading blocks, and thus shown as 2,800 cycles)

path / N	experim.	MOI	MB	MPH
Case 0	7100	7085	7085	7085
Case 1	2800	3379	3379	1150
Case 2	4200	4462	4462	1504
Case 3	820	640	640	229
Case 4	900	858	858	304
Case 5	3200	3557	3557	3557
Case 6	2600	2332	2393	2177
Case 7	1700	1590	1751	1453
Case 8	470	604	856	572
Case 9	660	604	856	572
Case 10	320	329	949	329
Case 11	1200	1073	2241	1073
Case 12	710	689	2023	689

as if 90º out of phase

as if proportional

The MOI method predicts that Cases 0 through 5 are proportional. This is reasonable, because the star and cross-shaped histories from Cases 1-4 are indeed the combination of 2 perpendicular proportional paths. These two perpendicular paths should not be considered as a single NP path, since they will most likely induce fatigue damage independently from each other in two perpendicular material planes. The MPH generates overly conservative predictions in these cases, since such enclosing surface method would not be able to distinguish e.g. between a cross-shaped and a circular history.

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For Cases 5-12, the MOI method also predicts the fatigue lives very well, agreeing with the MPH predictions. However, the MB method implicitly assumes that all 13 cases are proportional, leading to poor predictions for Cases 8-12.
The MOI method results in quite reasonable life predictions in all studied histories, within only 20% from the experimental results. Note that these are not curve fittings, they really are true predictions made using the MOI method (together with the SWT model for critical planes) without any adjustable parameter. The MPH method, on the other hand, gives poor life predictions for Cases 1-4, since it wrongfully assumes that these cross or star-shaped histories are 90^o out-of-phase, instead of being proportional (Table 1). And the MB method results in non-conservative predictions in Cases 8-12, since it wrongfully assumes that these paths are proportional.

4 CONCLUSIONS

A new method to calculate equivalent ranges was proposed, called the Moment Of Inertia (MOI) method. Since it is not based on path enclosures, it deals better with path shape dependence issues. Experimental results for 13 different multiaxial histories collected from comprehensive studies proved the effectiveness of the MOI method to predict the associated fatigue lives, when compared to the existing enclosing surface methods.

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